# BRICS Qualifying Exam

## Automation for Formal Verification: Exploiting Structure

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# Roadmap

Part I Automation in theorem proving: congruence closure

Part II Automation in model checking: hierarchical decomposition

Part III Automation in concurrency: identifying togetherness

# Roadmap

Part I Automation in theorem proving: congruence closure

- ⋆ congruence closure framework
- ⋆ bit-vector theories
- ⋆ computational complexity of solving

Part II Automation in model checking: hierarchical decomposition

Part III Automation in concurrency: identifying togetherness

# Automated Theorem Proving

$$\boxed{\Gamma \;\vdash\; \varphi}$$

$\Gamma$ : assumptions

$\varphi$ : conclusion

**Inference system** $\rightsquigarrow$ **Theorem prover**

**We want *reasonable* tool support:**

- high-level strategies (like *induct & simplify*)
- heuristics to search for proofs
- full automation in simple cases

# Automated Theorem Proving

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**Inference system** $\rightsquigarrow$ **Theorem prover**

sequent calculus — PVS

**We want *reasonable* tool support:**

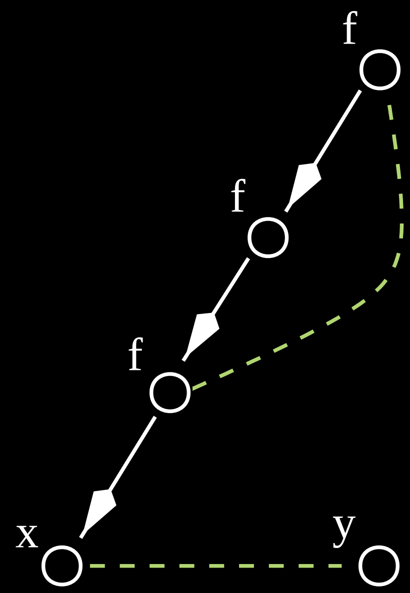
- high-level strategies (like *induct & simplify*) ✓
- heuristics to search for proofs ✓
- full automation in simple cases — congruence closure

# Encoding Equations in a DAG

**Given:** $f\,f\,f\,x = f\,x, \quad x = y$

**Conclude:** $f\,f\,f\,f\,f\,y = f\,x$

as a graph



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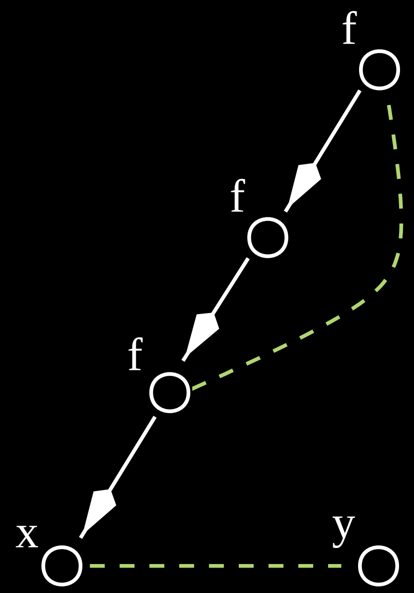
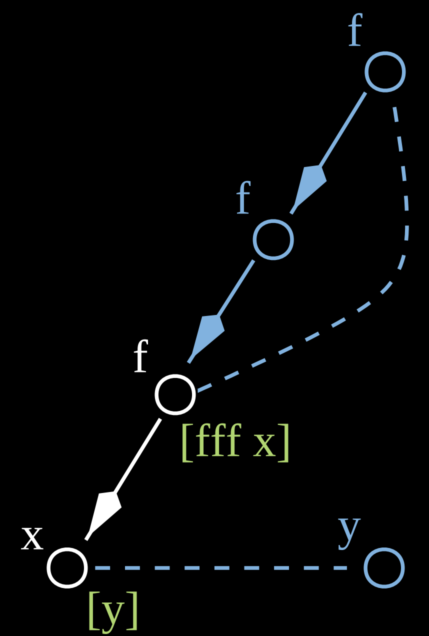



$\Longrightarrow$




cut out redundant parts

$$
\begin{aligned}
f\,f\,f\,f\,f\,y &= f\,f\,f\,f\,f\,x \\
f\,f\,f\,f\,f\,x &= f\,f\,f\,x \\
f\,f\,f\,x &= f\,x
\end{aligned}
$$

# Shostak Style Framework

**Theorem**

$$T \vdash a = b \quad \text{iff} \quad node(a)\ \hat{R}_T\ node(b)$$

Shostak '84: It is possible to combine *several* theories in this framework, if they are *algebraically solvable.*
For every theory we have

**canonizer:** term $t \longrightarrow \sigma(t)$

unique representation: $\models t = u \Leftrightarrow \sigma(t) \doteq \sigma(u)$

**solver:** equation $t = u \longrightarrow \bigwedge_i x_i = s_i$

explicit description of all solutions

# Bit-Vector Theories

## Core

- $x_{[n]} \ : \ bvec_n$
  number of bits $n$ a constant
- Constants $c_{[n]}$, $c \in I\!N$
- Concatenation:

$x_{[3]}$ $\quad$ $y_{[3]}$ $\quad$ $x_{[3]} \otimes y_{[3]}$

- Extraction:


$x_{[3]}$ $\quad$ $x_{[3]}[0:1]$

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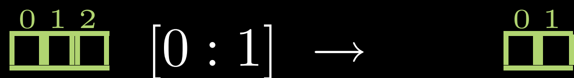



$x_{[3]}$ $x_{[3]}[0:1]$

## Extensions

- Boolean operations

$$x_{[3]} \wedge y_{[3]}$$

- Arithmetic (modulo $2^n$)

$$x_{[3]} +_{[3]} y_{[3]}$$

- Variable length

$$x_{[n]} \quad : \quad bvec_n$$
$$n \quad : \quad I\!N$$

- Variable Extraction

$$x_{[3]}[\mathrm{i{:}j}] \quad : \quad bvec_{j-i+1}$$
$$i, j \quad : \quad I\!N$$

# Canonizing Bit-Vector Terms

**Phase $\alpha$:** flatten extractions

$$\left( \overset{\boxed{0}\boxed{1}\boxed{2}}{x_{[3]}} \otimes \overset{\boxed{0}\boxed{1}\boxed{2}}{y_{[3]}} \right)[0:3] \quad \rightsquigarrow \quad \overset{\boxed{0}\boxed{1}\boxed{2}}{x_{[3]}} \otimes \overset{\boxed{0}}{y_{[3]}[0:0]}$$

**Phase $\beta$:** glue together

$$\overset{\boxed{0}\boxed{1}}{x_{[3]}[0:1]} \otimes \overset{\boxed{2}}{x_{[3]}[2:2]} \quad \rightsquigarrow \quad \overset{\boxed{0}\boxed{1}\boxed{2}}{x_{[3]}}$$

# Bit-Vector Solver: Chop

(1) $p_{[n]} \otimes t = q_{[n]} \otimes u$

$$\left\{ \begin{array}{l} p_{[n]} = q_{[n]} \\ t = u \end{array} \right\}$$

(1′) $p_{[n]} \otimes t = q_{[m]} \otimes u, \quad n < m$

$$\left\{ \begin{array}{l} p_{[n]} = \sigma(q_{[m]}[0 : n-1]) \\ \sigma(q_{[m]}[n : m-1]) \otimes u = t \\ q_{[m]} = \sigma(q_{[m]}[0 : n-1]) \otimes \sigma(q_{[m]}[n : m-1]) \end{array} \right\}$$

(1″) $p_{[n]} \otimes t = q_{[m]}$

$$\left\{ \begin{array}{l} p_{[n]} = \sigma(q_{[m]}[0 : n-1]) \\ \sigma(q_{[m]}[n : m-1]) = t \\ q_{[m]} = \sigma(q_{[m]}[0 : n-1]) \otimes \sigma(q_{[m]}[n : m-1]) \end{array} \right\}$$

# Bit-Vector Solver: Propagate

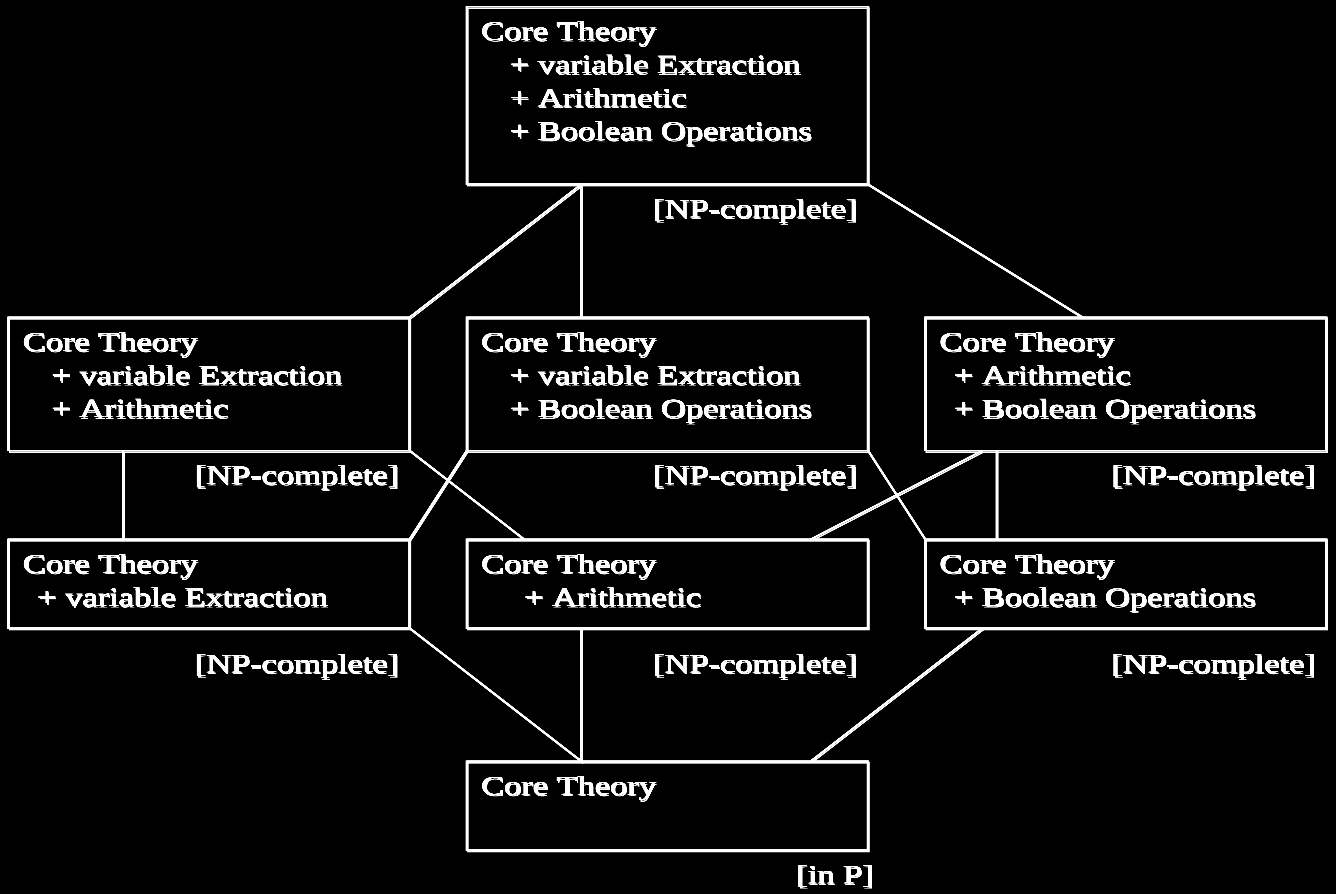
| | | | |
|---|---|---|---|
| (2) | $c = d$ , | $c \mathrel{\dot{\neq}} d$ | **FAIL** |
| (3) | $t = t$ | | $\{\}$ |
| (4) | $\left\{ \begin{array}{c} p = t \\ q = u \end{array} \right\}$ , | $q \preceq t$ | $\left\{ \begin{array}{l} p = t[q/u] \\ q = u \end{array} \right\}$ |
| (5) | $\left\{ \begin{array}{c} p = q \\ q = r \end{array} \right\}$ | | $\left\{ \begin{array}{c} p = r \\ q = r \end{array} \right\}$ |
| (6) | $\left\{ \begin{array}{c} p = q \\ q = p \end{array} \right\}$ | | $\{\ p = q\ \}$ |
| (7) | $\left\{ \begin{array}{c} p = t \\ p = u \end{array} \right\}$ | | $\left\{ \begin{array}{c} p = t \\ u = t \end{array} \right\}$ |
| (8) | $c = t,$ | $t \notin \mathcal{C}$ | $\{\ t = c\ \}$ |
| (9) | $p = q \otimes t,$ | $p \mathrel{\dot{\neq}} \sigma(q \otimes t)$ | $\{\ q \otimes t = p\ \}$ |

# Complexity of the Satisfiability Problem

Core Theory

[in P]

# Complexity of the Satisfiability Problem



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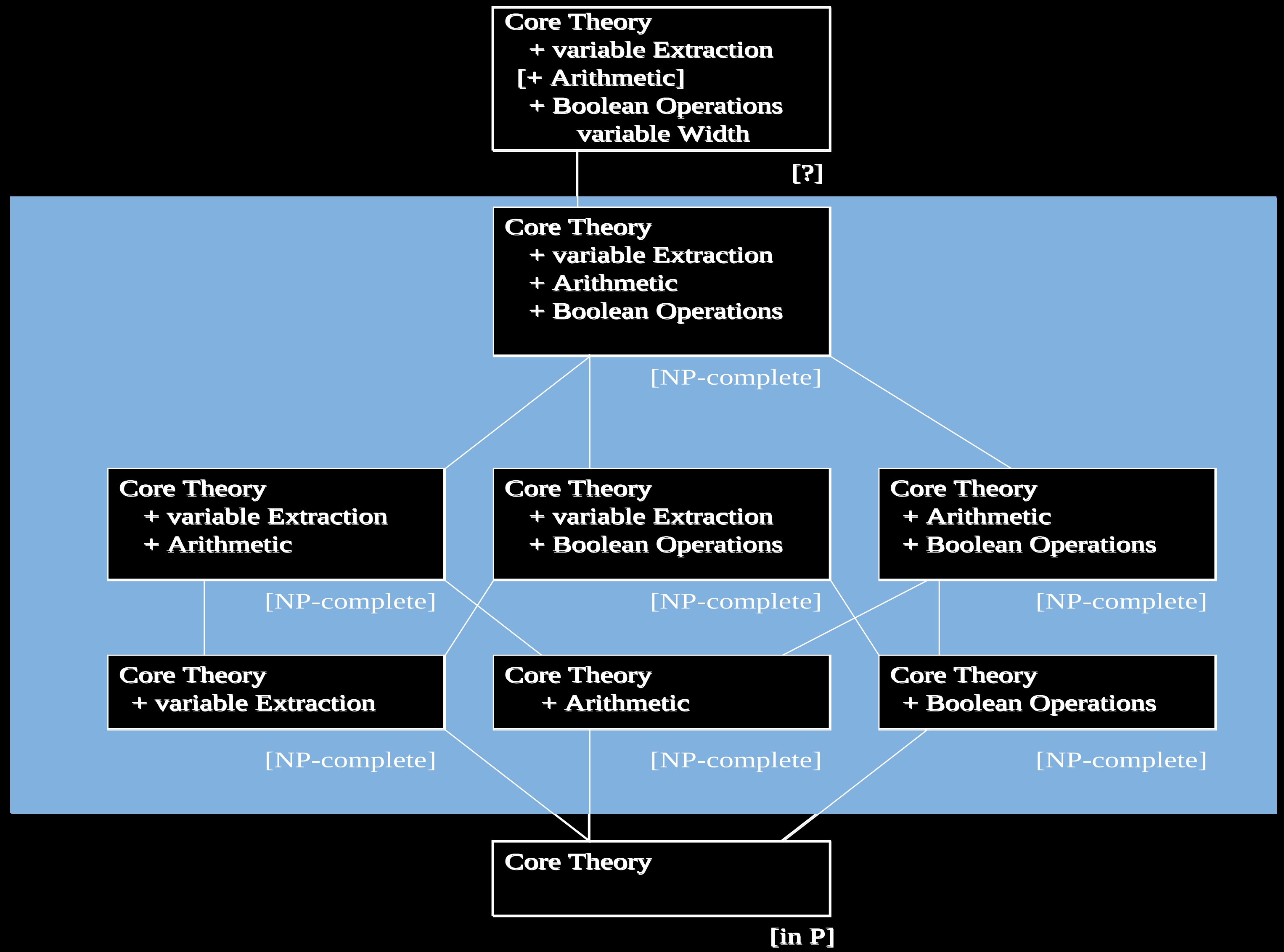
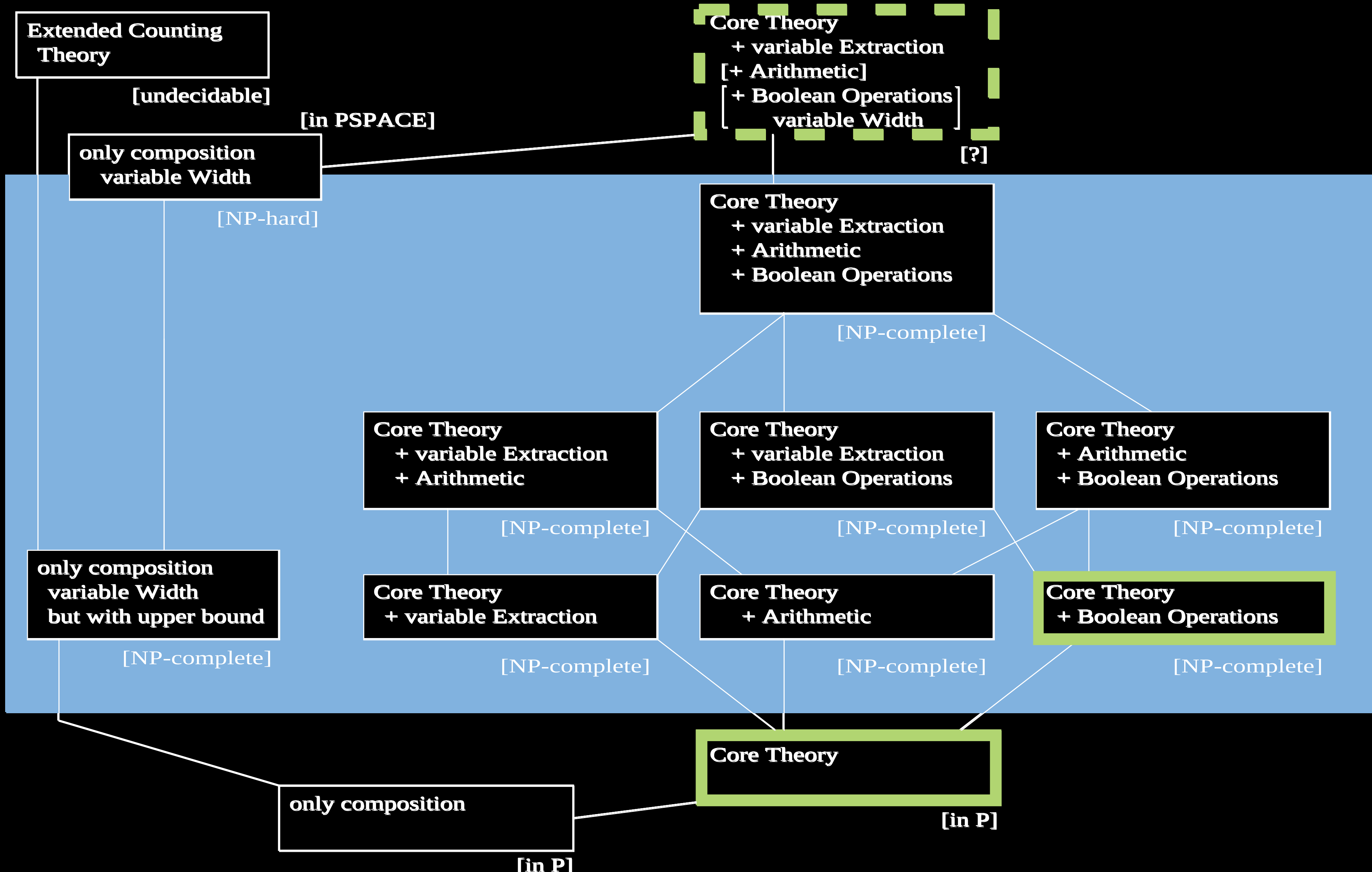


# Complexity of the Satisfiability Problem



# Expressiveness of Solving

$$\forall x.\exists y.\exists z.(x \vee y) \leftrightarrow \neg z$$

$$solve\big((x \vee y) \leftrightarrow \neg z\big) \quad = \quad \left\{ \begin{array}{ccc} x & = & a \\ y & = & b \\ z & = & \neg a \wedge \neg b \end{array} \right\}$$

$$\forall x.\exists y.\exists z.(x \vee y) \leftrightarrow \neg z \quad : \quad \textit{true}$$

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| | | | | | |
|---|---|---|---|---|---|
| $\forall x.\forall y.\forall z.(x \vee y) \leftrightarrow \neg z$ | : | *false* | $\exists x.\forall y.\forall z.(x \vee y) \leftrightarrow \neg z$ | : | *false* |
| $\forall x.\forall y.\exists z.(x \vee y) \leftrightarrow \neg z$ | : | *true* | $\exists x.\forall y.\exists z.(x \vee y) \leftrightarrow \neg z$ | : | *true* |
| $\forall x.\exists y.\forall z.(x \vee y) \leftrightarrow \neg z$ | : | *false* | $\exists x.\exists y.\forall z.(x \vee y) \leftrightarrow \neg z$ | : | *false* |
| $\forall x.\exists y.\exists z.(x \vee y) \leftrightarrow \neg z$ | : | *true* | $\exists x.\exists y.\exists z.(x \vee y) \leftrightarrow \neg z$ | : | *true* |

# Quantification Lemma

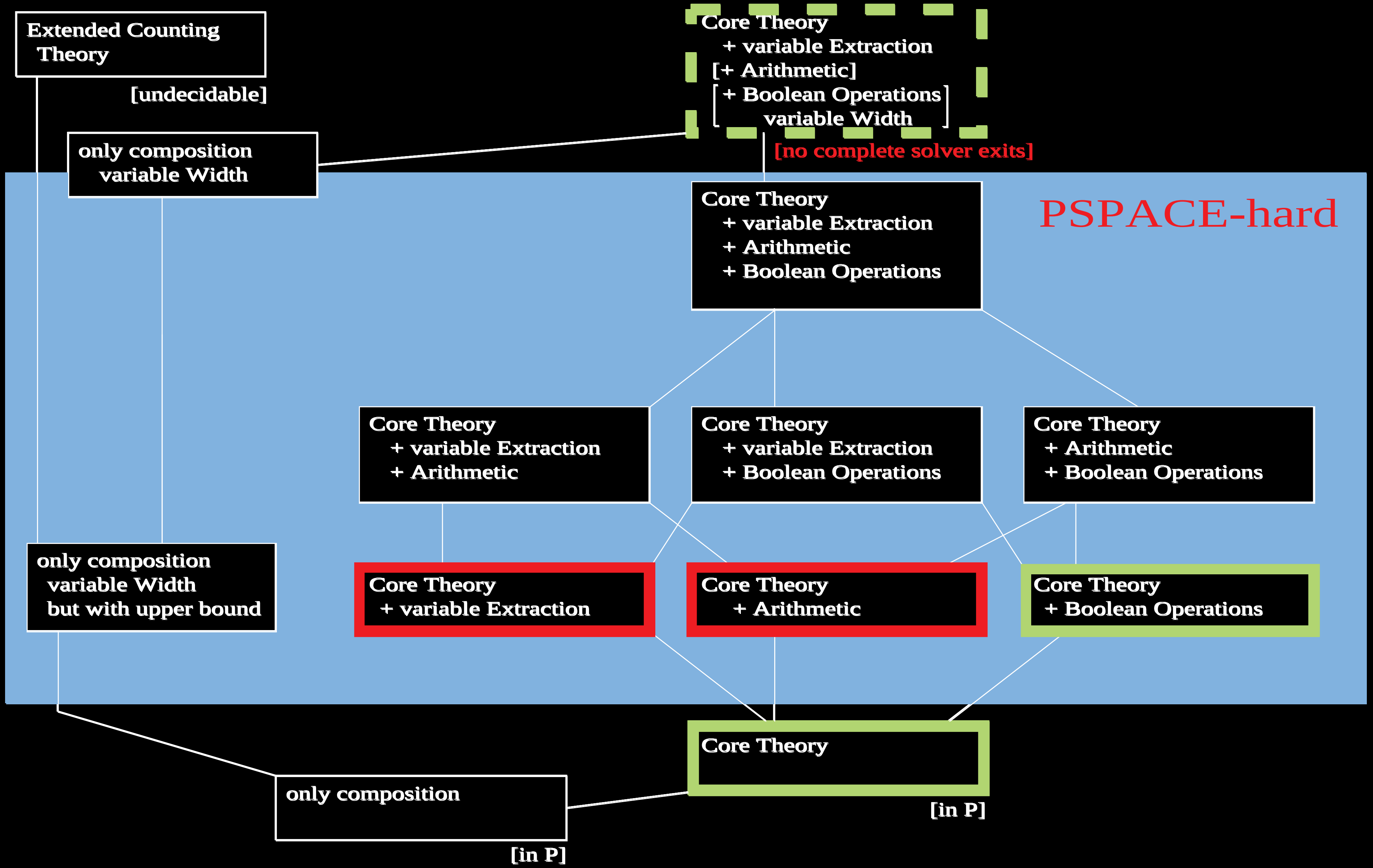
**General Method**

- start with a quantified equation $Q_1 x_1 \ldots Q_n x_n.\ t = u$
- let $S = \mathit{solve}(t = u)$;
  derive $\bigwedge_i x_i = s_i$ where
  - all $x_i$ appear in the right order
  - terms $s_i$ do not refer to $x_j$ with $j > i$
- check for $\forall$-quantified $x_i$, whether $s_i$ is *unrestricted*

small overhead in the size of the solution $\left( \mathcal{O}(|S| \cdot \log |S|) \right)$

$\Rightarrow$ Solving essentially decides all quantified versions

# Complexity Revisited



# Summary on Part I

**We established**

- canonizer and solver for the core theory
- two extensions:
  boolean operations (canonizer + solver)
  variable width (solver)
- quantification lemma

**Further development?**

- high computational complexity
- interesting fragments?
  $\Rightarrow$ algebraically solvability a serious restriction
- extension of the framework?
  $\Rightarrow$ seems to boil down to a case-split

# Roadmap

Part I Automation in theorem proving: congruence closure

- ⋆ congruence closure framework
- ⋆ bit-vector theories
- ⋆ computational complexity of solving

Part II Automation in model checking: hierarchical decomposition

- ⋆ model checking and temporal logic
- ⋆ a technique for temporal scaling
- ⋆ hierarchical structuring

Part III Automation in concurrency: identifying sequential parts

# Model Checking

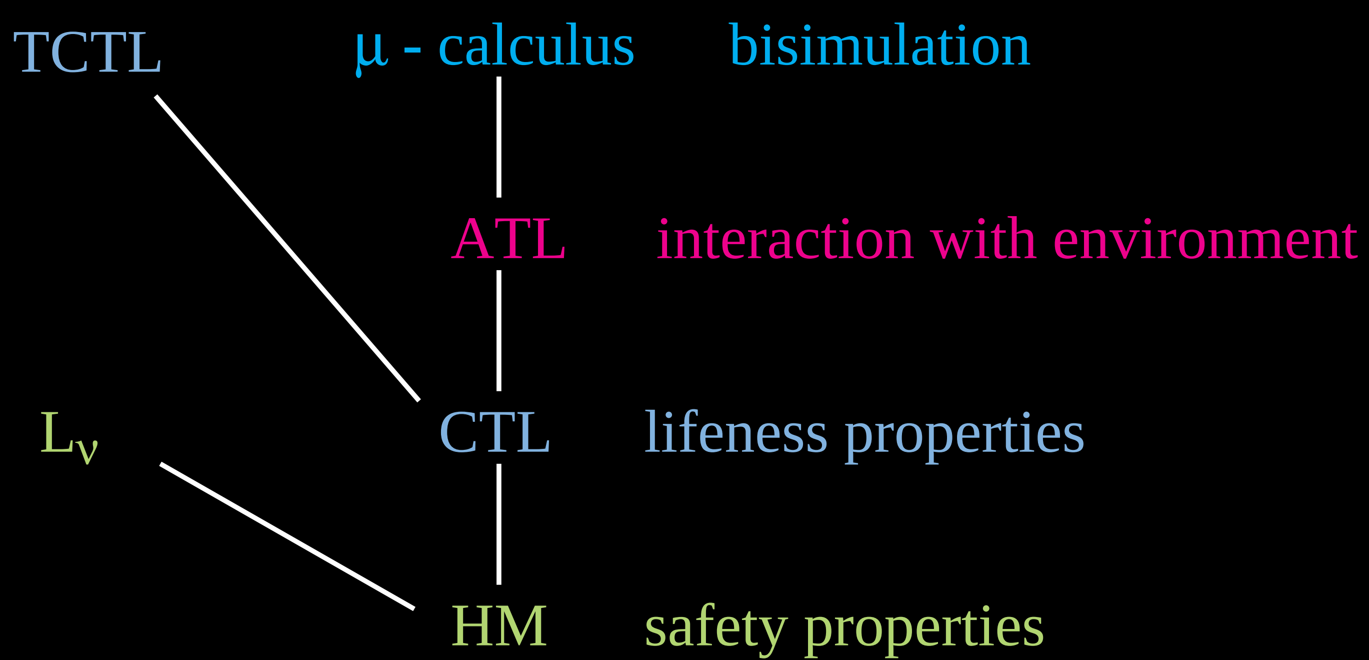
$$\boxed{M \models \varphi}$$

$M$ : description of the system

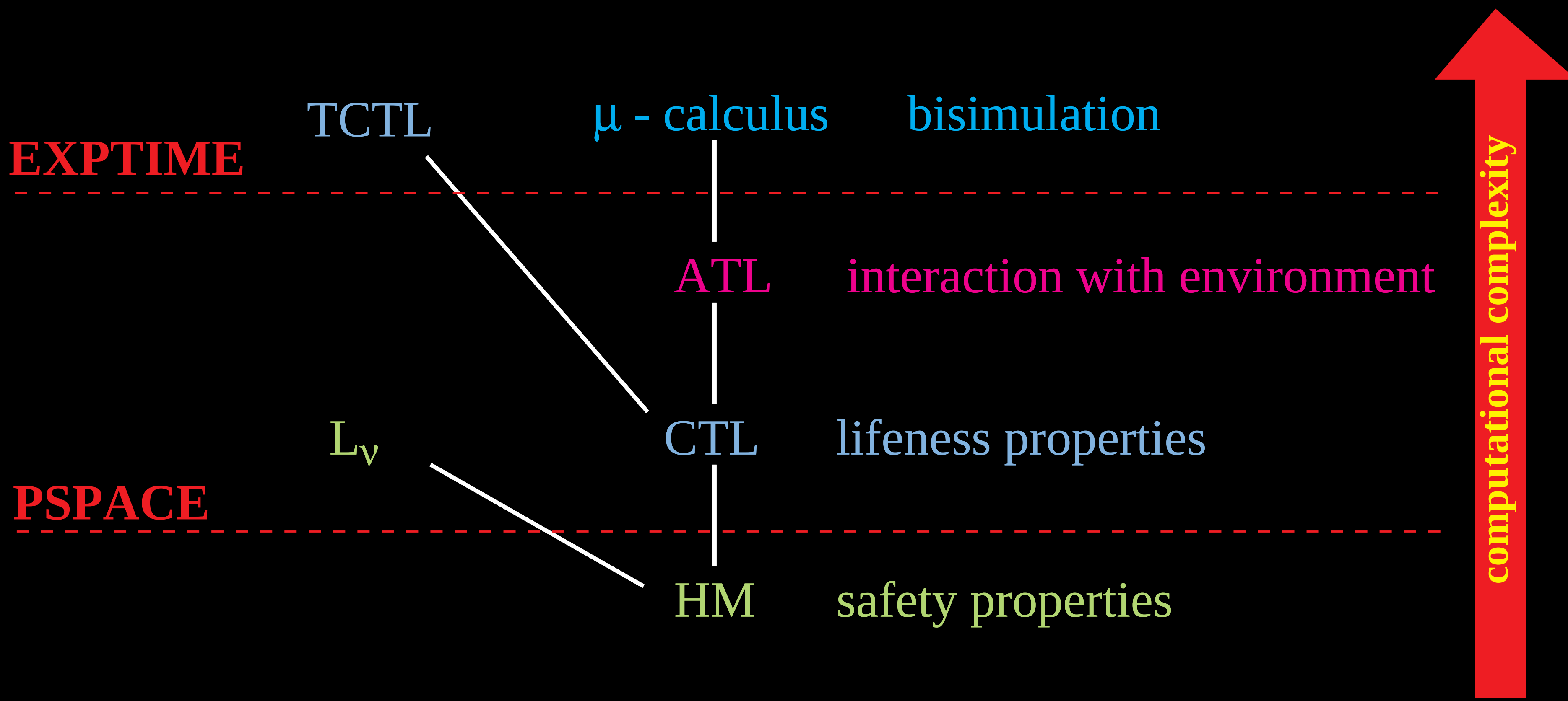
$\varphi$ : desired property

- easier than proving a general theorem
- completely automatic ('yes' or counterexample)
- efficient algorithms tailored for classes of problems

# Temporal Logics



# Temporal Logics



# Problems with Model Checking

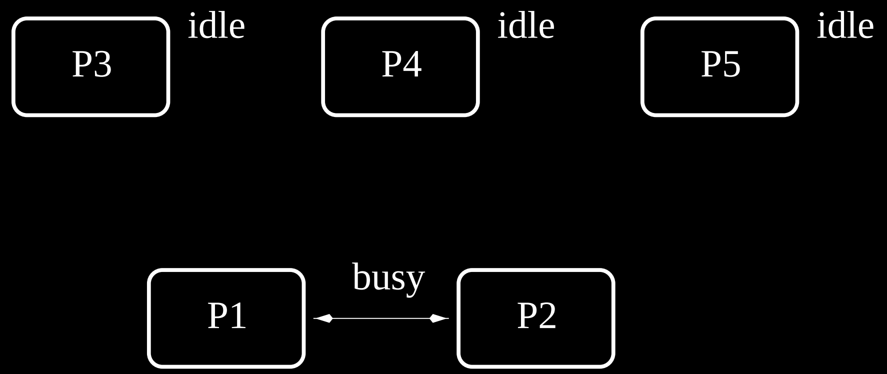
- exponential growth of the state space
  $\Rightarrow$ *state explosion problem*

- restricted expressiveness of the model
  $\Rightarrow$ experience required to make apt abstractions

- restricted expressiveness of the logics
  $\Rightarrow$ properties often have to be *encoded*

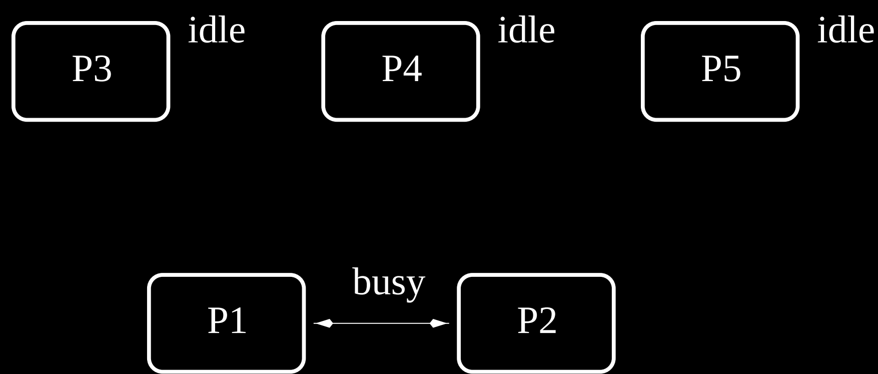
- excessive time/space consumption
  $\Rightarrow$ heuristics to make *state space exploration* more efficient

# Temporal Scaling

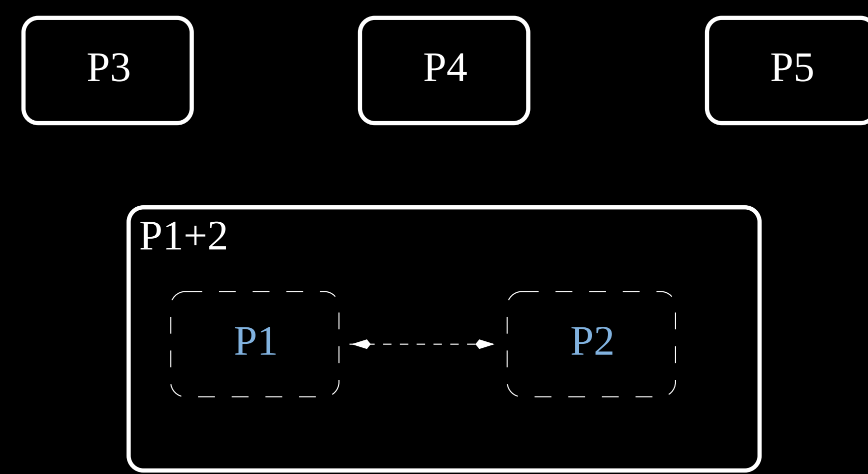



$$P1 \,||\, P2 \,||\, P3 \,||\, P4 \,||\, P5$$

# Temporal Scaling




$\Rightarrow$




Instead of:

$$P1 \,||\, P2 \,||\, P3 \,||\, P4 \,||\, P5$$

Use:

$$\mathit{hide\ busy\ in}\ \boxed{P1 \,||\, P2} \,||\, P3 \,||\, P4 \,||\, P5$$

# Model Checking with MOCHA

- parallel execution of components
- communication via shared variables (1 write/multi read)
- *ATL* model checker
- invariant check:
  allows temporal scaling via "next" $\Theta$ for $P$
  (*Rajeev Alur and Bow-Yaw Wang, CONCUR'99*)
- heuristic to preprocess a system for "next"

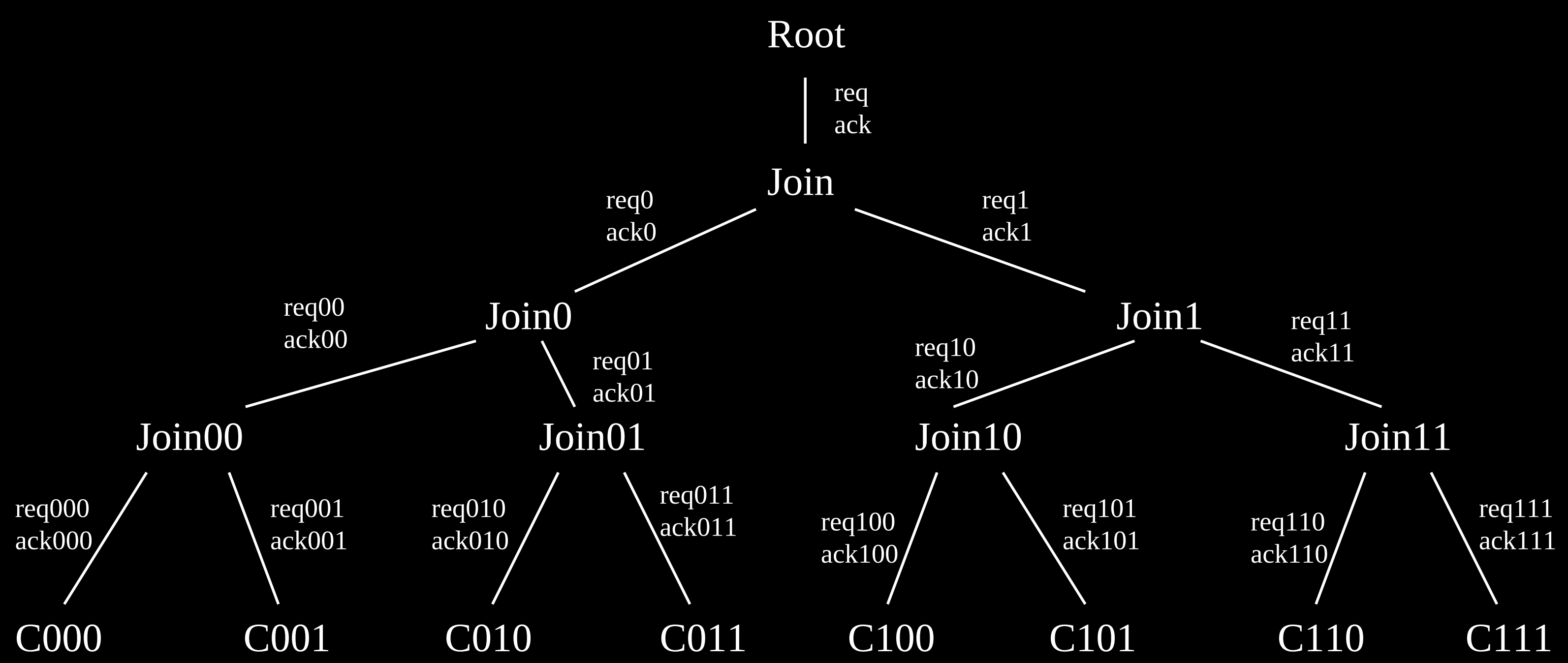
# Incremental Clustering

**input:** hypergraph $\mathcal{H} = (\mathcal{C}, \mathcal{E})$

**output:** *forest* with leaves $\mathcal{C}$

*PriorityQueue* $Q$
*forest* := { }
FORALL considered candidates $\mathcal{A} \subseteq \mathcal{C}$
    *insert*($\mathcal{A}$, $Q$)
WHILE *notempty*($Q$)
    $\mathcal{A}$ := *top*($Q$)
    fresh node Ⓐ
    *forest* := *forest* + ( Ⓐ $\mapsto$ $\mathcal{A}$ )
    redirect hyperedges: $A_i \in \mathcal{A}$ becomes Ⓐ
    FORALL $\mathcal{B} \in Q$ with $\mathcal{B} \cap \mathcal{A} \neq \emptyset$
        *remove*($\mathcal{B}$, $Q$)
    FORALL new candidates $\mathcal{D}$ containing Ⓐ
        *insert*($\mathcal{D}$, $Q$)

# Asynchronous Parity Computer

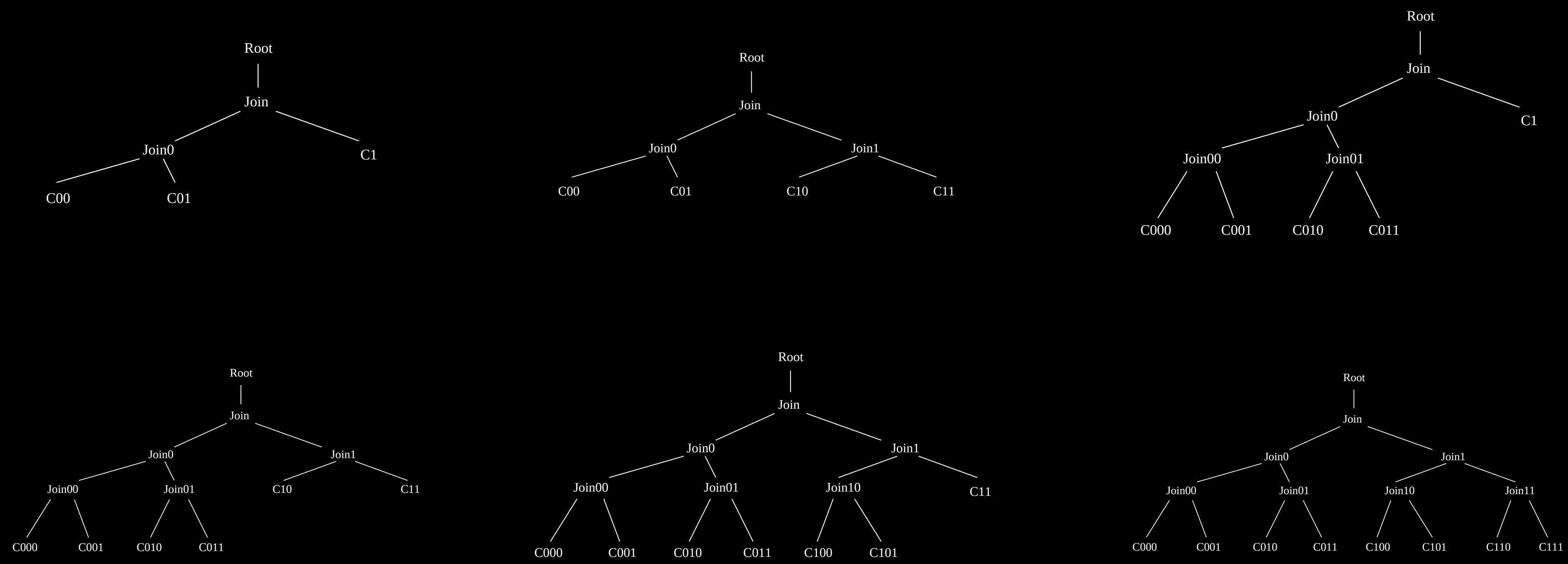



**Clients** : issue value *true* or *false*

**Joins** : compute *xor*

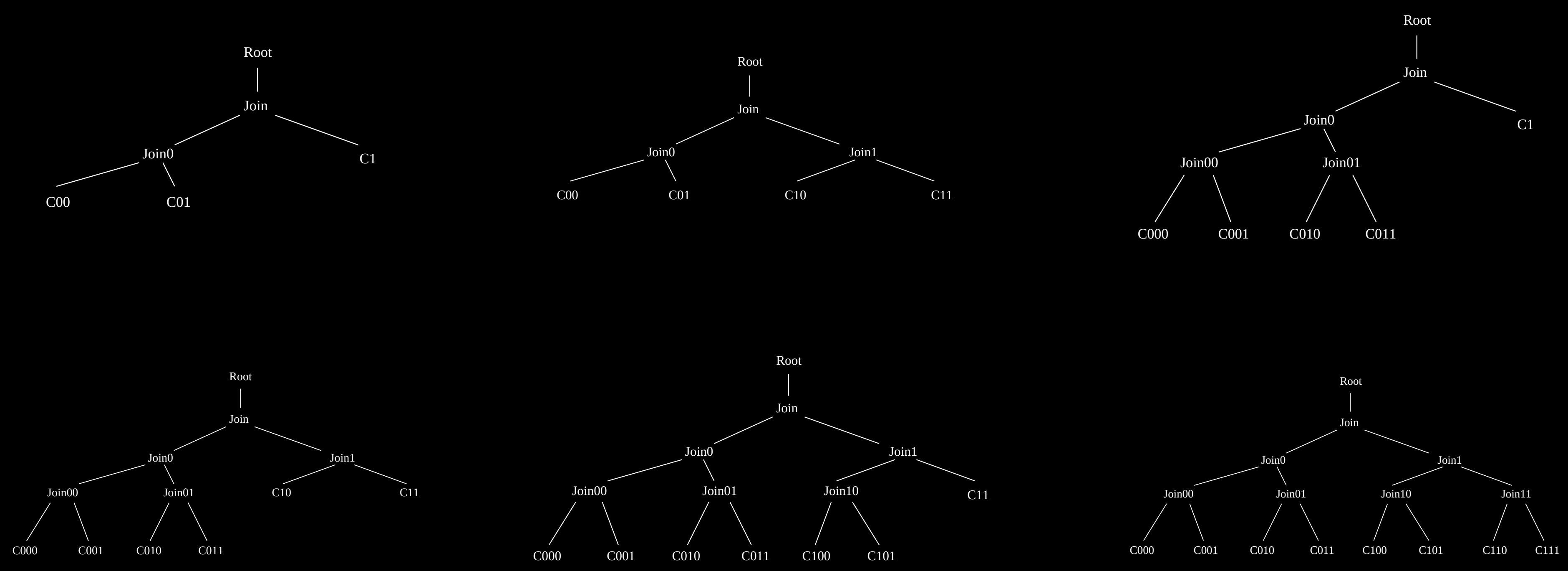
**Root** : acknowledges

# Good Heuristic Function




Cover-Number Size Egdes Depth

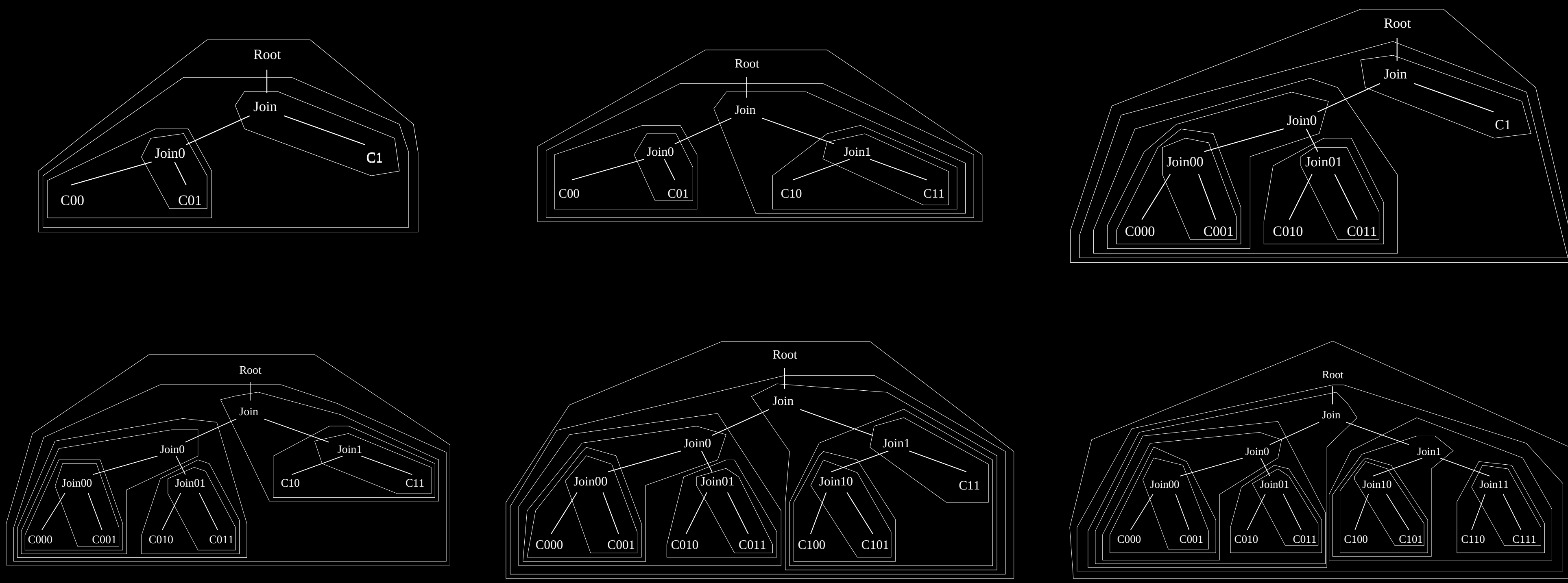
# Good Heuristic Function




$$\mathbf{r}^{+}_{pref}(\mathcal{A}) := \frac{\left|\{e \in \mathcal{E} \,\middle|\, e \subseteq \mathcal{A}\}\right|}{|\mathcal{A}|^2} + \frac{\varepsilon_1}{\left|\{e \in \mathcal{E} \,\middle|\, e \cap \mathcal{A} \neq \emptyset\}\right|} + \frac{\varepsilon_2}{depth(\mathcal{A})}$$

Cover-Number Size Egdes Depth

# Good Heuristic Function

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Cover-Number Size Egdes Depth

# Summary on Part II

**We established**

- incremental method for hierarchical decomposition
  (has direct application with temporal scaling)
- heuristic function based on 4 criteria:
  Cover-Number, Size, Edges, Depth
- sample problem, where this heuristic is well-behaved

**Left to do**

- validate method/function with other sample topologies
- investigate applicability beyond invariant properties
- (possibly) integration in the next Mocha release

# Guiding Idea

System:

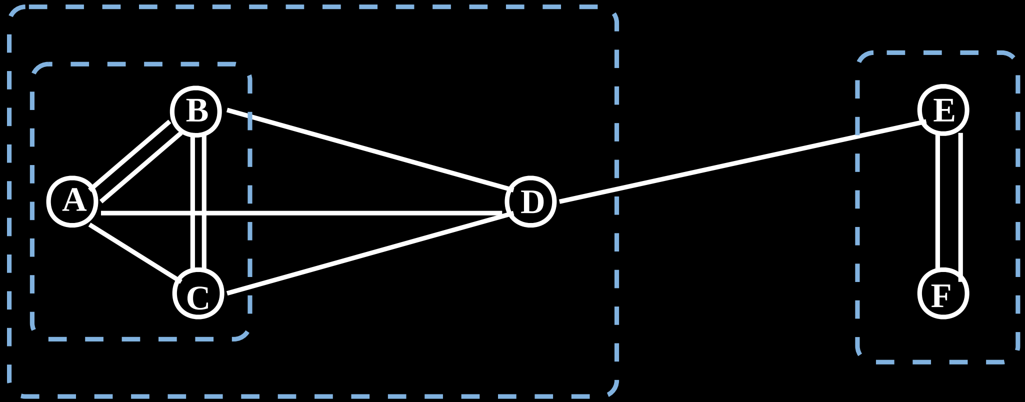
parallel components

Connections:

influence behavior

Structure:

often has *asymmetries*




{A,B,C} belong *strongly* together

{D, E} are not as "together" as {E, F}

**General Observation:**

If we can adapt our analysis to asymmetries, it becomes either

- more expressive

or • more efficient

| We aim to describe, detect and exploit "Togetherness" |
|---|

# Describe Togetherness

**Desired Characterization:**

$$\{\text{A, B}\} \text{ are together [to extend } \alpha] \quad \Leftrightarrow \quad ?$$

**Likely:** the links influence this

**Unlikely:** link information suffices

**Methodologies:**

- Identification by categorical models
- $CCS$- or $CSP$-like frameworks
- Properties derived from automata interaction

$\Rightarrow$ If goal too ambitious, restrict to special cases

# Detect Togetherness

**To be expected:** Good characterizations are computationally expensive

$\Rightarrow$ identify typical patterns

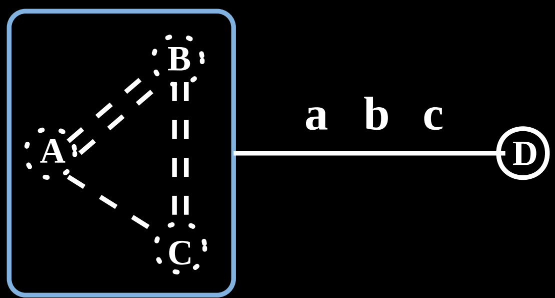
If infeasible, investigate *why*

$\Rightarrow$ lower bounds / hardness results

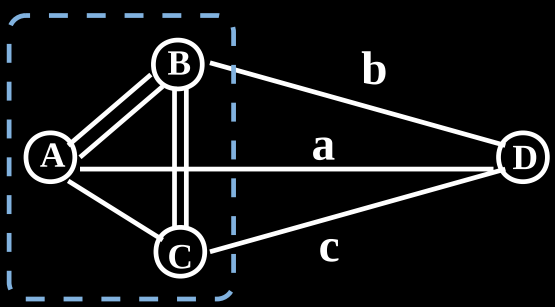
Reasonable hope for *partial* solutions

- develop heuristics (guided search)
- describe special cases

# Exploit Togetherness

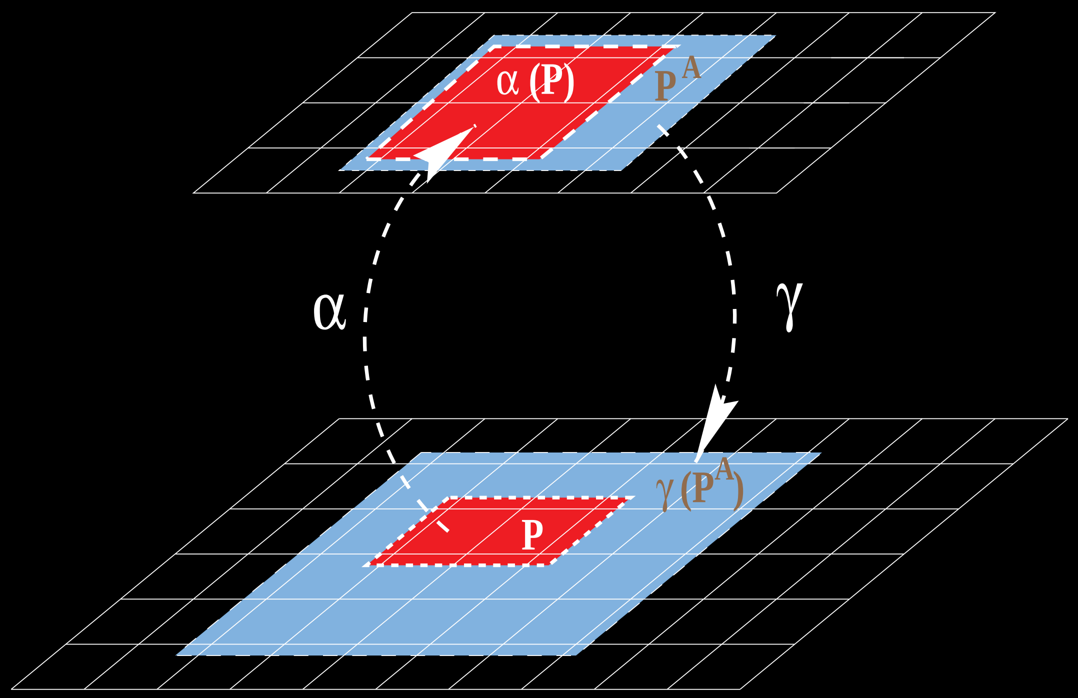



Abstraction




Given System

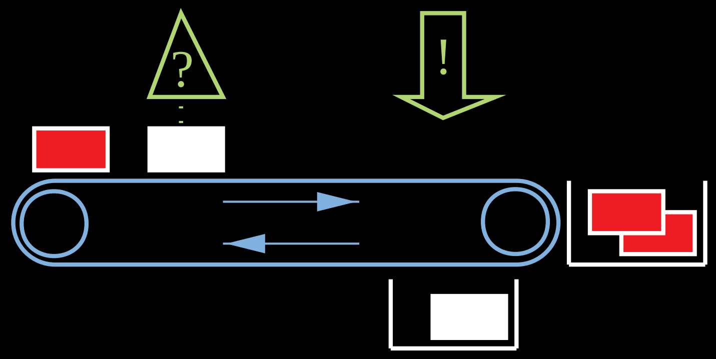
Galois Connection




$$\forall P, P^A :\ \alpha(P) \subseteq P^A \Leftrightarrow P \subseteq \gamma(P^A)$$

$\boxed{\text{Property holds in abstraction} \Longrightarrow \text{Property holds in actual system}}$

# Starting Point: Brick Sorter




Verify controller, running on LEGO® RCX

- Specified as timed automaton (UPPAAL)
- Proven Property: *always will kick off black*

**Problem**

model checking slow: many delay steps without real progress

**Possible solutions**

- temporal scaling: $\approx$ '*next* $\Theta$ *for* $P$'
- abbreviations: $1000\,a \;\longrightarrow\; 512\,a + 256\,a + 128\,a + 64\,a + 32\,a + 8\,a$
- building abstraction and prove it correct

# Summary

**Previous work**

- Bit-Vectors: 2 canonizers, 3 solvers, 2 new complexity results
- Model-Checking: heuristic to partition a system hierarchically

backed with (experimental) implementations

**Future plans**

- Develop a variation of 'next', tailored for timed automata
- Explore the phenomenon of *Togetherness* in concurrent systems in greater detail